

Minimization and Maximization of Empirical Risks: Applications in Data Unlearning and Decentralized Learning

Yaiza Bermudez

joint work with Samir Perlaza (Inria) and Iñaki Esnaola (University of Sheffield)

Selected Lectures on Statistical Machine Learning
February 25, 2026, Sophia Antipolis, France



Learning System and Notation



Figure: Learning system

The *training dataset* $\mathbf{z} \in \mathcal{Z}^n \triangleq (\mathcal{X} \times \mathcal{Y})^n$:

$$\mathbf{z} \triangleq ((x_1, y_1), \dots, (x_n, y_n)) \in \mathcal{Z}^n,$$

with size $n \in \mathbb{N}$ data points.

Learning System and Notation



Figure: Learning system

The *training dataset* $\mathbf{z} \in \mathcal{Z}^n \triangleq (\mathcal{X} \times \mathcal{Y})^n$:

$$\mathbf{z} \triangleq ((x_1, y_1), \dots, (x_n, y_n)) \in \mathcal{Z}^n,$$

with size $n \in \mathbb{N}$ data points.

Loss function: $\ell : \mathcal{M} \times \mathcal{Z} \rightarrow [0, +\infty)$.

Learning System and Notation



Figure: Learning system

The *training dataset* $\mathbf{z} \in \mathcal{Z}^n \triangleq (\mathcal{X} \times \mathcal{Y})^n$:

$$\mathbf{z} \triangleq ((x_1, y_1), \dots, (x_n, y_n)) \in \mathcal{Z}^n,$$

with size $n \in \mathbb{N}$ data points.

Loss function: $\ell : \mathcal{M} \times \mathcal{Z} \rightarrow [0, +\infty)$.

Empirical risk:

$$\mathbf{L} : \begin{cases} \mathcal{Z}^n \times \mathcal{M} \longrightarrow [0, +\infty) \\ (\mathbf{z}, \boldsymbol{\theta}) \longmapsto \frac{1}{n} \sum_{i=1}^n \ell(\boldsymbol{\theta}, x_i, y_i), \end{cases}$$

Empirical Risk Optimizations

Let $\lambda > 0$,

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q)$$

Let $\lambda < 0$,

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q)$$

Samir M. Perlaza, Gaetan Bisson, Iñaki Esnaola, Alain Jean-Marie, and Stefano Rini "Empirical Risk Minimization with Relative Entropy Regularization". IEEE Transactions on Information Theory, vol. 70, no. 7, pp. 5122 – 5161, July, 2024.

Samir M. Perlaza and Gaetan Bisson. "Variations on the expectation due to changes in the probability measure". Entropy, 27(8:865):1–20, Aug. 2025

Definition

Given a σ -finite measure $Q \in \Delta(\mathcal{M})$; and a $\lambda \in \mathbb{R} \setminus \{0\}$, the probability measure $P_{\boldsymbol{\theta}|\mathcal{Z}}^{(Q,\lambda)} \in \Delta(\mathcal{M}|\mathcal{Z}^n)$ is said to be an (L, Q, λ) -Gibbs conditional probability measure if for all $\mathbf{z} \in \mathcal{S}$, $K_{Q,\mathbf{z}}\left(-\frac{1}{\lambda}\right) < +\infty$; for some set $\mathcal{S} \subseteq \mathcal{Z}^n$; and for all $(\mathbf{z}, \boldsymbol{\theta}) \in \mathcal{S} \times \text{supp } Q$,

$$\frac{dP_{\boldsymbol{\theta}|\mathbf{z}=\mathbf{z}}^{(Q,\lambda)}}{dQ}(\boldsymbol{\theta}) = \exp\left(-\frac{1}{\lambda}L(\mathbf{z}, \boldsymbol{\theta}) - K_{Q,\mathbf{z}}\left(-\frac{1}{\lambda}\right)\right),$$

where the function $K_{Q,\mathbf{z}}$ is defined as follows

$$K_{Q,\mathbf{z}} : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ t \longmapsto \log\left(\int \exp(tL(\mathbf{z}, \boldsymbol{\theta})) dQ(\boldsymbol{\theta})\right). \end{cases}$$

Empirical Risk Optimization

Let $\lambda > 0$,

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q)$$

Let $\lambda < 0$,

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q)$$

Samir M. Perlaza, Gaetan Bisson, Iñaki Esnaola, Alain Jean-Marie, and Stefano Rini "Empirical Risk Minimization with Relative Entropy Regularization". IEEE Transactions on Information Theory, vol. 70, no. 7, pp. 5122 – 5161, July, 2024.

Samir M. Perlaza and Gaetan Bisson. "Variations on the expectation due to changes in the probability measure". Entropy, 27(8:865):1–20, Aug. 2025

Empirical Risk Optimization

Let $\lambda > 0$,

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q)$$

Let $\lambda > 0$ be chosen such that $D(P_{\theta|Z=z}^{(Q, \lambda)} \parallel Q) = \gamma$,

$$\begin{aligned} \min_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel Q) \leq \gamma. \end{aligned}$$

Let $\lambda < 0$,

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q)$$

Let $\lambda < 0$ be chosen such that $D(P_{\theta|Z=z}^{(Q, \lambda)} \parallel Q) = \gamma$,

$$\begin{aligned} \max_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel Q) \leq \gamma. \end{aligned}$$

Samir M. Perlaza, Gaetan Bisson, Iñaki Esnaola, Alain Jean-Marie, and Stefano Rini "Empirical Risk Minimization with Relative Entropy Regularization". IEEE Transactions on Information Theory, vol. 70, no. 7, pp. 5122 – 5161, July, 2024.

Samir M. Perlaza and Gaetan Bisson. "Variations on the expectation due to changes in the probability measure". Entropy, 27(8:865):1–20, Aug. 2025

Geometric Representation

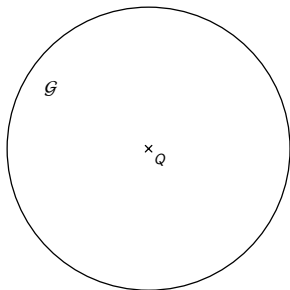
$$\left\{P_{\Theta|Z=z}^{(Q,\lambda)}\right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q), & \text{if } \lambda > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q), & \text{if } \lambda < 0 \end{cases}$$

Samir M. Perlaza, Gaetan Bisson, Iñaki Esnaola, Alain Jean-Marie, and Stefano Rini "Empirical Risk Minimization with Relative Entropy Regularization". IEEE Transactions on Information Theory, vol. 70, no. 7, pp. 5122 – 5161, July, 2024.

Samir M. Perlaza and Gaetan Bisson. "Variations on the expectation due to changes in the probability measure". Entropy, 27(8:865):1–20, Aug. 2025

Geometric Representation

$$\left\{P_{\Theta|Z=z}^{(Q,\lambda)}\right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q), & \text{if } \lambda > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q), & \text{if } \lambda < 0 \end{cases}$$



$$\mathcal{G} = \left\{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq D\left(P_{\Theta|Z=z}^{(Q,\lambda)} \parallel Q\right)\right\}.$$

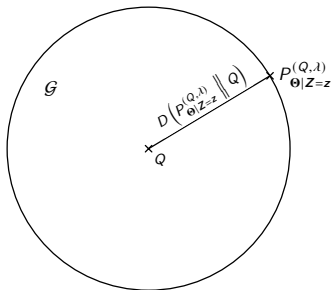
$$\left\{P_{\Theta|Z=z}^{(Q,\lambda)}\right\} = \begin{cases} \arg \text{opt}_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel Q) \leq \gamma \end{cases}$$

Samir M. Perlaza, Gaetan Bisson, Iñaki Esnaola, Alain Jean-Marie, and Stefano Rini "Empirical Risk Minimization with Relative Entropy Regularization". IEEE Transactions on Information Theory, vol. 70, no. 7, pp. 5122 – 5161, July, 2024.

Samir M. Perlaza and Gaetan Bisson. "Variations on the expectation due to changes in the probability measure". Entropy, 27(8:865):1–20, Aug. 2025

Geometric Representation

$$\left\{P_{\Theta|Z=z}^{(Q,\lambda)}\right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q), & \text{if } \lambda > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) + \lambda D(P \parallel Q), & \text{if } \lambda < 0 \end{cases}$$



$$\mathcal{G} = \left\{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq D\left(P_{\Theta|Z=z}^{(Q,\lambda)} \parallel Q\right)\right\}.$$

$$\left\{P_{\Theta|Z=z}^{(Q,\lambda)}\right\} = \begin{cases} \arg \text{opt}_{P \in \Delta_Q(\mathcal{M})} \int L(z, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel Q) \leq \gamma \end{cases}$$

Samir M. Perlaza, Gaetan Bisson, Iñaki Esnaola, Alain Jean-Marie, and Stefano Rini "Empirical Risk Minimization with Relative Entropy Regularization". IEEE Transactions on Information Theory, vol. 70, no. 7, pp. 5122 – 5161, July, 2024.

Samir M. Perlaza and Gaetan Bisson. "Variations on the expectation due to changes in the probability measure". Entropy, 27(8:865):1–20, Aug. 2025

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda D(P \parallel Q), \text{ if } \lambda > 0;$$

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda D(P \parallel Q), \text{ if } \lambda < 0.$$

- Choice of λ ? [Bu-2024]
- Choice of Q ?

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda D(P \parallel Q), \text{ if } \lambda > 0;$$

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda D(P \parallel Q), \text{ if } \lambda < 0.$$

What happens if Q is a Gibbs measure ?

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda D(P \parallel Q), \text{ if } \lambda > 0;$$

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda D(P \parallel Q), \text{ if } \lambda < 0.$$

Let $\mathbf{z}_1 \in \mathcal{Z}^{n_1}$ be the training dataset; let $\mathbf{z}_2 \in \mathcal{Z}^{n_2}$ be another datasets and let $\mathbf{z}_0 = (\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{Z}^{n_1} \times \mathcal{Z}^{n_2}$. Consider some $\alpha > 0$, $\lambda_0 > 0$ and $\lambda_2 > 0$.

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=z_1}^{(Q, \alpha)}\right), \text{ if } \lambda_1 > 0;$$

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=z_1}^{(Q, \alpha)}\right), \text{ if } \lambda_1 < 0.$$

Let $z_1 \in \mathcal{Z}^{n_1}$ be the training dataset; let $z_2 \in \mathcal{Z}^{n_2}$ be another datasets and let $z_0 = (z_1, z_2) \in \mathcal{Z}^{n_1} \times \mathcal{Z}^{n_2}$. Consider some $\alpha > 0$, $\lambda_0 > 0$ and $\lambda_2 > 0$.

CASE A: prior based on the training dataset z_1 , i.e. $P_{\Theta|Z=z_1}^{(Q, \alpha)}$;

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=z_0}^{(Q, \lambda_0)}\right), \text{ if } \lambda_1 > 0;$$

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=z_0}^{(Q, \lambda_0)}\right), \text{ if } \lambda_1 < 0.$$

Let $z_1 \in \mathcal{Z}^{n_1}$ be the training dataset; let $z_2 \in \mathcal{Z}^{n_2}$ be another datasets and let $z_0 = (z_1, z_2) \in \mathcal{Z}^{n_1} \times \mathcal{Z}^{n_2}$. Consider some $\alpha > 0$, $\lambda_0 > 0$ and $\lambda_2 > 0$.

CASE A: prior based on the training dataset z_1 , i.e. $P_{\Theta|Z=z_1}^{(Q, \alpha)}$;

CASE B: prior based on the training dataset z_1 and the dataset z_2 , i.e. $P_{\Theta|Z=z_0}^{(Q, \lambda_0)}$;

$$\min_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=z_2}^{(Q, \lambda_2)}\right), \text{ if } \lambda_1 > 0;$$

$$\max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=z_2}^{(Q, \lambda_2)}\right), \text{ if } \lambda_1 < 0.$$

Let $\mathbf{z}_1 \in \mathcal{Z}^{n_1}$ be the training dataset; let $\mathbf{z}_2 \in \mathcal{Z}^{n_2}$ be another datasets and let $\mathbf{z}_0 = (\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{Z}^{n_1} \times \mathcal{Z}^{n_2}$. Consider some $\alpha > 0$, $\lambda_0 > 0$ and $\lambda_2 > 0$.

CASE A: prior based on the training dataset $\mathbf{z}_1$, i.e. $P_{\Theta|Z=\mathbf{z}_1}^{(Q, \alpha)}$;

CASE B: prior based on the training dataset $\mathbf{z}_1$ and the dataset $\mathbf{z}_2$, i.e. $P_{\Theta|Z=\mathbf{z}_0}^{(Q, \lambda_0)}$;

CASE C: prior based on the dataset $\mathbf{z}_2$, i.e. $P_{\Theta|Z=\mathbf{z}_2}^{(Q, \lambda)}$.

CASE A: prior based on training data $\mathbf{z}_1$

$$\begin{aligned} & \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=\mathbf{z}_1}^{(Q, \alpha)}\right), \text{ if } \lambda_1 > 0; \\ & \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=\mathbf{z}_1}^{(Q, \alpha)}\right), \text{ if } \lambda_1 < 0. \end{aligned}$$

Let $\mathbf{z}_1 \in \mathcal{Z}^{n_1}$ be the training dataset; let $\mathbf{z}_2 \in \mathcal{Z}^{n_2}$ be another datasets and let $\mathbf{z}_0 = (\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{Z}^{n_1} \times \mathcal{Z}^{n_2}$. Consider some $\alpha > 0$, $\lambda_0 > 0$ and $\lambda_2 > 0$.

CASE A: prior based on the training dataset $\mathbf{z}_1$, i.e. $P_{\Theta|Z=\mathbf{z}_1}^{(Q, \alpha)}$;

CASE B: prior based on the training dataset $\mathbf{z}_1$ and the dataset $\mathbf{z}_2$, i.e. $P_{\Theta|Z=\mathbf{z}_0}^{(Q, \lambda_0)}$;

CASE C: prior based on the dataset $\mathbf{z}_2$, i.e. $P_{\Theta|Z=\mathbf{z}_2}^{(Q, \lambda_2)}$.

CASE A: Prior Based on the Training Dataset $\mathbf{z}_1$

$$\left\{ P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)} \right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}\right), & \text{if } \lambda_1 > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}\right), & \text{if } \lambda_1 < 0 \end{cases}$$

CASE A: Prior Based on the Training Dataset $\mathbf{z}_1$

$$\left\{ P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)} \right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}\right), & \text{if } \lambda_1 > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}\right), & \text{if } \lambda_1 < 0 \end{cases}$$

Theorem

The $(L, P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)$ -Gibbs probability measure $P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)}$, with $\alpha \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \in \mathbb{R} \setminus \{0\}$, satisfies for all $\boldsymbol{\theta} \in \text{supp } Q$,

$$\frac{dP_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(-\left(\left(\frac{1}{\lambda_1} + \frac{1}{\alpha}\right)L(\mathbf{z}_1, \boldsymbol{\theta})\right)\right)}{\int \exp\left(-\left(\left(\frac{1}{\lambda_1} + \frac{1}{\alpha}\right)L(\mathbf{z}_1, \nu)\right)\right) dQ(\nu)}$$

CASE A: Prior Based on the Training Dataset $\mathbf{z}_1$

$$\left\{ P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)} \right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}\right), & \text{if } \lambda_1 > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}\right), & \text{if } \lambda_1 < 0 \end{cases}$$

Theorem

The $(L, P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)$ -Gibbs probability measure $P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)}$, with $\alpha \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \in \mathbb{R} \setminus \{0\}$, satisfies for all $\boldsymbol{\theta} \in \text{supp } Q$,

$$\frac{dP_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}, \lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(-\left(\left(\frac{1}{\lambda_1} + \frac{1}{\alpha}\right)L(\mathbf{z}_1, \boldsymbol{\theta})\right)\right)}{\int \exp\left(-\left(\left(\frac{1}{\lambda_1} + \frac{1}{\alpha}\right)L(\mathbf{z}_1, \nu)\right)\right) dQ(\nu)} = \frac{dP_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q, \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\alpha}})}}{dQ}(\boldsymbol{\theta}).$$

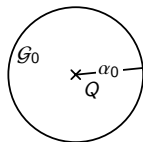
CASE A: Geometrical Representation

Let Q be a σ -finite measure and let $\alpha > 0$.

$$\left\{ P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha),\lambda_1} \right\} = \left\{ \begin{array}{l} \arg \text{opt}_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) \, dP(\boldsymbol{\theta}) \\ \text{s.t. } D(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)}) \leq \alpha_2 \end{array} \right.$$

$$\alpha_0 \triangleq D\left(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q,\alpha)} \parallel Q\right);$$

$$\mathcal{G}_0 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0\};$$



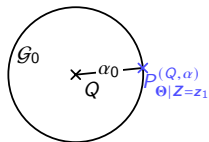
CASE A: Geometrical Representation

Let Q be a σ -finite measure and let $\alpha > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \right\} = \left\{ \begin{array}{l} \arg \text{opt}_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)}) \leq \alpha_2 \end{array} \right.$$

$$\alpha_0 \triangleq D\left(P_{\Theta|Z=z_1}^{(Q,\alpha)} \parallel Q\right);$$

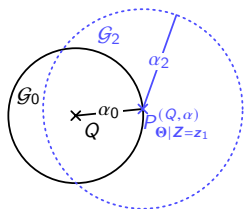
$$\mathcal{G}_0 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0\};$$



CASE A: Geometrical Representation

Let Q be a σ -finite measure and let $\alpha > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \right\} = \left\{ \begin{array}{l} \arg \text{opt}_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)}) \leq \alpha_2 \end{array} \right.$$



$$\alpha_0 \triangleq D \left(P_{\Theta|Z=z_1}^{(Q,\alpha)} \parallel Q \right);$$

$$\mathcal{G}_0 \triangleq \{ P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0 \};$$

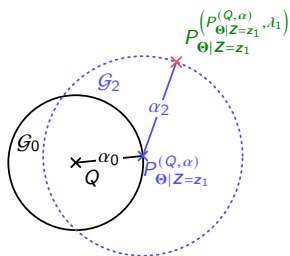
$$\alpha_2 \triangleq D \left(P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)} \right);$$

$$\mathcal{G}_2 \triangleq \left\{ P \in \Delta_Q(\mathcal{M}) : D(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)}) \leq \alpha_2 \right\};$$

CASE A: Geometrical Representation

Let Q be a σ -finite measure and let $\alpha > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \right\} = \left\{ \begin{array}{l} \arg \text{opt}_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)}) \leq \alpha_2 \end{array} \right.$$



$$\alpha_0 \triangleq D \left(P_{\Theta|Z=z_1}^{(Q,\alpha)} \parallel Q \right);$$

$$\mathcal{G}_0 \triangleq \{ P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0 \};$$

$$\alpha_2 \triangleq D \left(P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)} \right);$$

$$\mathcal{G}_2 \triangleq \left\{ P \in \Delta_Q(\mathcal{M}) : D \left(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)} \right) \leq \alpha_2 \right\};$$

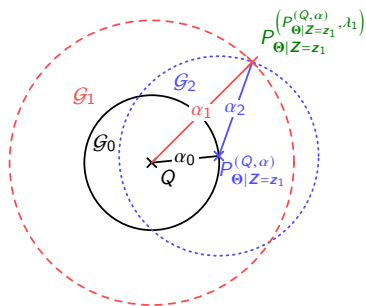
$$\alpha_1 \triangleq D \left(P_{\Theta|Z=z_1}^{(Q, \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\alpha}})} \parallel Q \right);$$

$$\mathcal{G}_1 \triangleq \{ P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_1 \}.$$

CASE A: Geometrical Representation

Let Q be a σ -finite measure and let $\alpha > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \right\} = \left\{ \begin{array}{l} \arg \text{opt}_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)}) \leq \alpha_2 \end{array} \right.$$



$$\alpha_0 \triangleq D \left(P_{\Theta|Z=z_1}^{(Q,\alpha)} \parallel Q \right);$$

$$\mathcal{G}_0 \triangleq \{ P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0 \};$$

$$\alpha_2 \triangleq D \left(P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)} \right);$$

$$\mathcal{G}_2 \triangleq \left\{ P \in \Delta_Q(\mathcal{M}) : D \left(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)} \right) \leq \alpha_2 \right\};$$

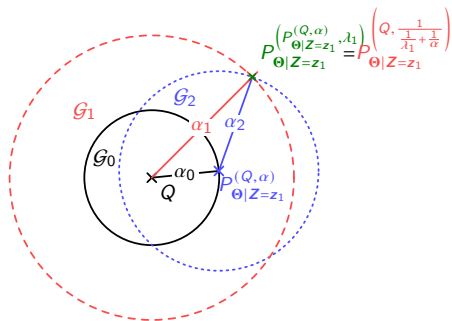
$$\alpha_1 \triangleq D \left(P_{\Theta|Z=z_1}^{(Q, \frac{1}{\lambda_1} + \frac{1}{\alpha})} \parallel Q \right);$$

$$\mathcal{G}_1 \triangleq \{ P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_1 \}.$$

CASE A: Geometrical Representation

Let Q be a σ -finite measure and let $\alpha > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \right\} = \left\{ \begin{array}{l} \arg \text{opt}_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)}) \leq \alpha_2 \end{array} \right.$$



$$\alpha_0 \triangleq D \left(P_{\Theta|Z=z_1}^{(Q,\alpha)} \parallel Q \right);$$

$$\mathcal{G}_0 \triangleq \{ P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0 \};$$

$$\alpha_2 \triangleq D \left(P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_1}^{(Q,\alpha)}, \lambda_1)} \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)} \right);$$

$$\mathcal{G}_2 \triangleq \left\{ P \in \Delta_Q(\mathcal{M}) : D \left(P \parallel P_{\Theta|Z=z_1}^{(Q,\alpha)} \right) \leq \alpha_2 \right\};$$

$$\alpha_1 \triangleq D \left(P_{\Theta|Z=z_1}^{(Q, \frac{1}{\lambda_1} + \frac{1}{\alpha})} \parallel Q \right);$$

$$\mathcal{G}_1 \triangleq \{ P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_1 \}.$$

CASE B: Prior Based on the Training Dataset $\mathbf{z}_1$ and the Dataset $\mathbf{z}_2$

$$\begin{aligned} & \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q, \lambda_0)}\right), \text{ if } \lambda_0 > 0; \\ & \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q, \lambda_0)}\right), \text{ if } \lambda_0 < 0. \end{aligned}$$

Let $\mathbf{z}_1 \in \mathcal{Z}^{n_1}$ be the training dataset; let $\mathbf{z}_2 \in \mathcal{Z}^{n_2}$ be another datasets and let $\mathbf{z}_0 = (\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{Z}^{n_1} \times \mathcal{Z}^{n_2}$. Consider some $\alpha > 0$, $\lambda_0 > 0$ and $\lambda_2 > 0$.

CASE A: prior based on the training dataset $\mathbf{z}_1$, i.e. $P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q, \alpha)}$;

CASE B: prior based on the training dataset $\mathbf{z}_1$ and the dataset $\mathbf{z}_2$, i.e. $P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q, \lambda_0)}$;

CASE C: prior based on the dataset $\mathbf{z}_2$, i.e. $P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q, \lambda_2)}$.

CASE B: Prior Based on the Training Dataset $\mathbf{z}_1$ and the Dataset $\mathbf{z}_2$

$$\left\{ P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}, \lambda_1)} \right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}\right), & \text{if } \lambda_1 > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}\right), & \text{if } \lambda_1 < 0 \end{cases}$$

CASE B: Prior Based on the Training Dataset $\mathbf{z}_1$ and the Dataset $\mathbf{z}_2$

$$\left\{ P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)} \right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}\right), & \text{if } \lambda_1 > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}\right), & \text{if } \lambda_1 < 0 \end{cases}$$

Theorem

The $(L, P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}, \lambda_1)$ -Gibbs probability measure $P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}$, with $\lambda_0 > 0$ and $\lambda_1 \in \mathbb{R} \setminus \{0\}$, satisfies for all $\boldsymbol{\theta} \in \text{supp } Q$,

$$\frac{dP_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) L(\mathbf{z}_1, \boldsymbol{\theta}) + \frac{n_2}{n_0} L(\mathbf{z}_2, \boldsymbol{\theta}) \right)\right)}{\int \exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) L(\mathbf{z}_1, \boldsymbol{\nu}) + \frac{n_2}{n_0} L(\mathbf{z}_2, \boldsymbol{\nu}) \right)\right) dQ(\boldsymbol{\nu})}.$$

CASE B: Prior Based on the Training Dataset $\mathbf{z}_1$ and the Dataset $\mathbf{z}_2$

$$\begin{aligned} \left\{ P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)} \right\} &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \frac{n_2}{n_0} \int L(\mathbf{z}_2, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q) \\ &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \frac{\lambda_0}{\lambda_1} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \int L(\mathbf{z}_0, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q) \end{aligned}$$

$$\frac{dP_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) L(\mathbf{z}_1, \boldsymbol{\theta}) + \frac{n_2}{n_0} L(\mathbf{z}_2, \boldsymbol{\theta}) \right)\right)}{\int \exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) L(\mathbf{z}_1, \boldsymbol{\nu}) + \frac{n_2}{n_0} L(\mathbf{z}_2, \boldsymbol{\nu}) \right)\right) dQ(\boldsymbol{\nu})}.$$

CASE B: Prior Based on the Training Dataset $\mathbf{z}_1$ and the Dataset $\mathbf{z}_2$

$$\begin{aligned} \left\{ P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}, \lambda_1)} \right\} &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \frac{n_2}{n_0} \int L(\mathbf{z}_2, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q) \\ &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \frac{\lambda_0}{\lambda_1} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \int L(\mathbf{z}_0, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q) \end{aligned}$$

$$\frac{dP_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}, \lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) L(\mathbf{z}_1, \boldsymbol{\theta}) + \frac{n_2}{n_0} L(\mathbf{z}_2, \boldsymbol{\theta}) \right)\right)}{\int \exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) L(\mathbf{z}_1, \boldsymbol{\nu}) + \frac{n_2}{n_0} L(\mathbf{z}_2, \boldsymbol{\nu}) \right)\right) dQ(\boldsymbol{\nu})}.$$

Changes in the contribution of the dataset $\mathbf{z}_1$ in the training:

- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0} \right) > 0$, up-weights the contribution;

CASE B: Prior Based on the Training Dataset $\mathbf{z}_1$ and the Dataset $\mathbf{z}_2$

$$\begin{aligned}\left\{P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}\right\} &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \frac{n_2}{n_0} \int L(\mathbf{z}_2, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q) \\ &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \frac{\lambda_0}{\lambda_1} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \int L(\mathbf{z}_0, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q)\end{aligned}$$

$$\frac{dP_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right)L(\mathbf{z}_1, \boldsymbol{\theta}) + \frac{n_2}{n_0}L(\mathbf{z}_2, \boldsymbol{\theta})\right)\right)}{\int \exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right)L(\mathbf{z}_1, \boldsymbol{\nu}) + \frac{n_2}{n_0}L(\mathbf{z}_2, \boldsymbol{\nu})\right)\right) dQ(\boldsymbol{\nu})}.$$

Changes in the contribution of the dataset $\mathbf{z}_1$ in the training:

- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) > 0$, up-weights the contribution;
- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) < 0$, down-weights the contribution;

CASE B: Prior Based on the Training Dataset $\mathbf{z}_1$ and the Dataset $\mathbf{z}_2$

$$\begin{aligned}\left\{P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}\right\} &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \frac{n_2}{n_0} \int L(\mathbf{z}_2, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q) \\ &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \frac{\lambda_0}{\lambda_1} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \int L(\mathbf{z}_0, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q)\end{aligned}$$

$$\frac{dP_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right)L(\mathbf{z}_1, \boldsymbol{\theta}) + \frac{n_2}{n_0}L(\mathbf{z}_2, \boldsymbol{\theta})\right)\right)}{\int \exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right)L(\mathbf{z}_1, \boldsymbol{\nu}) + \frac{n_2}{n_0}L(\mathbf{z}_2, \boldsymbol{\nu})\right)\right) dQ(\boldsymbol{\nu})}.$$

Changes in the contribution of the dataset $\mathbf{z}_1$ in the training:

- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) > 0$, up-weights the contribution;
- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) < 0$, down-weights the contribution;
- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) = 0$, cancels the contribution.

CASE B: Prior Based on the Training Dataset $\mathbf{z}_1$ and the Dataset $\mathbf{z}_2$

$$\begin{aligned}\left\{P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}\right\} &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \frac{n_2}{n_0} \int L(\mathbf{z}_2, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q) \\ &= \arg \min_{P \in \Delta_Q(\mathcal{M})} \frac{\lambda_0}{\lambda_1} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \int L(\mathbf{z}_0, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_0 D(P \parallel Q)\end{aligned}$$

$$\frac{dP_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)},\lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right)L(\mathbf{z}_1, \boldsymbol{\theta}) + \frac{n_2}{n_0}L(\mathbf{z}_2, \boldsymbol{\theta})\right)\right)}{\int \exp\left(\frac{-1}{\lambda_0} \left(\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right)L(\mathbf{z}_1, \boldsymbol{\nu}) + \frac{n_2}{n_0}L(\mathbf{z}_2, \boldsymbol{\nu})\right)\right) dQ(\boldsymbol{\nu})}.$$

Changes in the contribution of the dataset $\mathbf{z}_1$ in the training:

- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) > 0$, up-weights the contribution;
- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) < 0$, down-weights the contribution;
- $\left(\frac{\lambda_0}{\lambda_1} + \frac{n_1}{n_0}\right) = 0$, cancels the contribution.

Exact Unlearning

$$\left\{ P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}, \lambda_1)} \right\} = \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\Theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q,\lambda_0)}\right), \text{ with } \lambda_1 < 0$$

Exact Unlearning

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, \lambda_1)} \right\} = \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) + \lambda_1 D\left(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right), \text{ with } \lambda_1 < 0$$

Definition (Exact unlearning)

Consider the algorithm represented by the (L, Q, λ_0) -Gibbs probability measure $P_{\Theta|Z=z_0}^{(Q,\lambda_0)}$ and the datasets z_1 and z_2 to be unlearned and retained, respectively.

An algorithm $P \in \Delta_Q(\mathcal{M})$ is said to achieve exact unlearning of z_1 from $P_{\Theta|Z=z_0}^{(Q,\lambda_0)}$, if for all $\theta \in \text{supp } Q$,

$$\frac{dP}{dP_{\Theta|Z=z_2}^{(Q,\lambda_2)}}(\theta) = 1,$$

for some $\lambda_2 > 0$.

Exact Unlearning

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \right\} = \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) + -\frac{n_0}{n_1}\lambda_0 D\left(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right),$$

Theorem

Assume that

$$\lambda_1 = -\frac{n_0}{n_1}\lambda_0 \text{ and } \lambda_2 = \frac{n_0}{n_2}\lambda_0,$$

for some $\lambda_0 > 0$. Then, the (L, Q, λ_2) -Gibbs probability measure $P_{\Theta|Z=z_2}^{(Q,\lambda_2)}$ and the $(L, P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, \lambda_1)$ -Gibbs probability measure $P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, \lambda_1)}$ satisfy for all $\theta \in \text{supp } Q$,

$$\frac{dP_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, \lambda_1)}}{dP_{\Theta|Z=z_2}^{(Q,\lambda_2)}}(\theta) = 1.$$

Geometric Representation of Unlearning

Let Q be a σ -finite measure and let $\lambda_0 > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \right\} = \left\{ \begin{array}{l} \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D\left(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right) \leq \alpha_2 \end{array} \right.$$

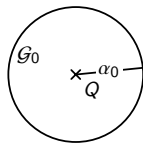
Geometric Representation of Unlearning

Let Q be a σ -finite measure and let $\lambda_0 > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \right\} = \left\{ \begin{array}{l} \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}) \leq \alpha_2 \end{array} \right.$$

$$\alpha_0 \triangleq D(P_{\Theta|Z=z_0}^{(Q,\lambda_0)} \parallel Q);$$

$$\mathcal{G}_0 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0\};$$



Geometric Representation of Unlearning

Let Q be a σ -finite measure and let $\lambda_0 > 0$.

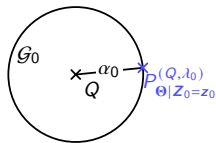
$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \right\} = \left\{ \begin{array}{l} \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}) \leq \alpha_2 \end{array} \right.$$

$$\alpha_0 \triangleq D(P_{\Theta|Z=z_0}^{(Q,\lambda_0)} \parallel Q);$$

$$\mathcal{G}_0 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0\};$$

$$\alpha_2 \triangleq D\left(P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right);$$

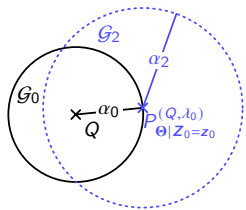
$$\mathcal{G}_2 \triangleq \left\{P \in \Delta_Q(\mathcal{M}) : D(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}) \leq \alpha_2\right\};$$



Geometric Representation of Unlearning

Let Q be a σ -finite measure and let $\lambda_0 > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \right\} = \left\{ \begin{array}{l} \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}) \leq \alpha_2 \end{array} \right.$$



$$\alpha_0 \triangleq D\left(P_{\Theta|Z=z_0}^{(Q,\lambda_0)} \parallel Q\right);$$

$$\mathcal{G}_0 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0\};$$

$$\alpha_2 \triangleq D\left(P_{\Theta|Z=z_0}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right);$$

$$\mathcal{G}_2 \triangleq \left\{P \in \Delta_Q(\mathcal{M}) : D\left(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right) \leq \alpha_2\right\};$$

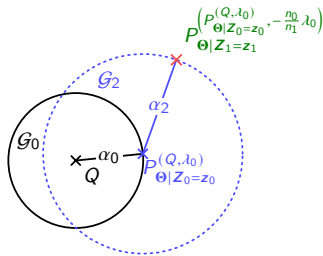
$$\alpha_1 \triangleq D\left(P_{\Theta|Z=z_2}^{(Q,\frac{n_0}{n_2}\lambda_0)} \parallel Q\right);$$

$$\mathcal{G}_1 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_1\};$$

Geometric Representation of Unlearning

Let Q be a σ -finite measure and let $\lambda_0 > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \right\} = \left\{ \begin{array}{l} \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}) \leq \alpha_2 \end{array} \right.$$



$$\alpha_0 \triangleq D\left(P_{\Theta|Z=z_0}^{(Q,\lambda_0)} \parallel Q\right);$$

$$\mathcal{G}_0 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0\};$$

$$\alpha_2 \triangleq D\left(P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right);$$

$$\mathcal{G}_2 \triangleq \left\{P \in \Delta_Q(\mathcal{M}) : D\left(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right) \leq \alpha_2\right\};$$

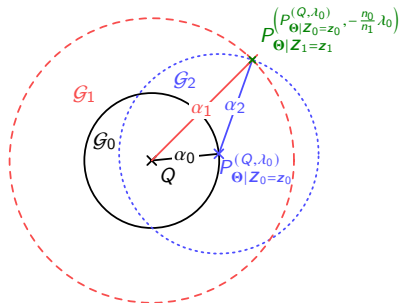
$$\alpha_1 \triangleq D\left(P_{\Theta|Z=z_2}^{(Q, \frac{n_0}{n_2}\lambda_0)} \parallel Q\right);$$

$$\mathcal{G}_1 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_1\};$$

Geometric Representation of Unlearning

Let Q be a σ -finite measure and let $\lambda_0 > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \right\} = \left\{ \begin{array}{l} \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}) \leq \alpha_2 \end{array} \right.$$



$$\alpha_0 \triangleq D\left(P_{\Theta|Z=z_0}^{(Q,\lambda_0)} \parallel Q\right);$$

$$\mathcal{G}_0 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0\};$$

$$\alpha_2 \triangleq D\left(P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right);$$

$$\mathcal{G}_2 \triangleq \left\{P \in \Delta_Q(\mathcal{M}) : D\left(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right) \leq \alpha_2\right\};$$

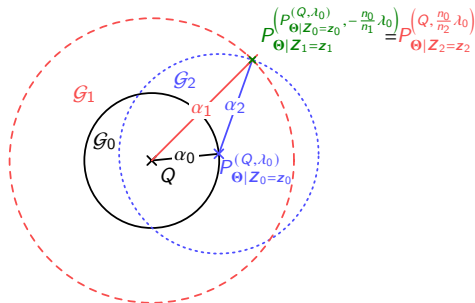
$$\alpha_1 \triangleq D\left(P_{\Theta|Z=z_2}^{(Q, \frac{n_0}{n_2}\lambda_0)} \parallel Q\right);$$

$$\mathcal{G}_1 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_1\};$$

Geometric Representation of Unlearning

Let Q be a σ -finite measure and let $\lambda_0 > 0$.

$$\left\{ P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \right\} = \left\{ \begin{array}{l} \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(z_1, \theta) dP(\theta) \\ \text{s.t. } D(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}) \leq \alpha_2 \end{array} \right.$$



$$\alpha_0 \triangleq D\left(P_{\Theta|Z=z_0}^{(Q,\lambda_0)} \parallel Q\right);$$

$$\mathcal{G}_0 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_0\};$$

$$\alpha_2 \triangleq D\left(P_{\Theta|Z=z_1}^{(P_{\Theta|Z=z_0}^{(Q,\lambda_0)}, -\frac{n_0}{n_1}\lambda_0)} \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right);$$

$$\mathcal{G}_2 \triangleq \left\{P \in \Delta_Q(\mathcal{M}) : D\left(P \parallel P_{\Theta|Z=z_0}^{(Q,\lambda_0)}\right) \leq \alpha_2\right\};$$

$$\alpha_1 \triangleq D\left(P_{\Theta|Z=z_2}^{(Q, \frac{n_0}{n_2}\lambda_0)} \parallel Q\right);$$

$$\mathcal{G}_1 \triangleq \{P \in \Delta_Q(\mathcal{M}) : D(P \parallel Q) \leq \alpha_1\};$$

CASE C: prior based on data $\mathbf{z}_2$

$$\begin{aligned} & \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q, \lambda_0)}\right), \text{ if } \lambda_2 > 0; \\ & \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q, \lambda_0)}\right), \text{ if } \lambda_2 < 0. \end{aligned}$$

Let $\mathbf{z}_1 \in \mathcal{Z}^{n_1}$ be the training dataset; let $\mathbf{z}_2 \in \mathcal{Z}^{n_2}$ be another datasets and let $\mathbf{z}_0 = (\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{Z}^{n_1} \times \mathcal{Z}^{n_2}$. Consider some $\alpha > 0$, $\lambda_0 > 0$ and $\lambda_2 > 0$.

CASE A: prior based on the training dataset $\mathbf{z}_1$, i.e. $P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(Q, \alpha)}$;

CASE B: prior based on the training dataset $\mathbf{z}_1$ and the dataset $\mathbf{z}_2$, i.e. $P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_0}^{(Q, \lambda_0)}$;

CASE C: prior based on the dataset $\mathbf{z}_2$, i.e. $P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q, \lambda_2)}$.

CASE C: Prior Based on the Dataset $\mathbf{z}_2$

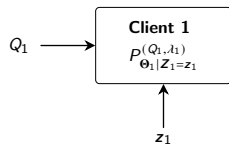
$$\left\{ P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q,\lambda_2)},\lambda_1)} \right\} = \begin{cases} \arg \min_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q,\lambda_2)}\right), & \text{if } \lambda_1 > 0; \\ \arg \max_{P \in \Delta_Q(\mathcal{M})} \int L(\mathbf{z}_1, \boldsymbol{\theta}) dP(\boldsymbol{\theta}) + \lambda_1 D\left(P \parallel P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q,\lambda_2)}\right), & \text{if } \lambda_1 < 0 \end{cases}$$

Theorem

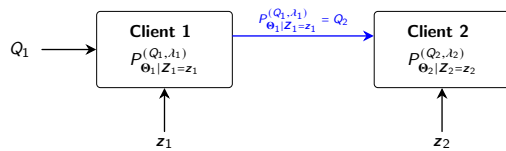
The $(L, P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q,\lambda_2)}, \lambda_1)$ -Gibbs probability measure $P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q,\lambda_2)},\lambda_1)}$, with $\lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \in \mathbb{R} \setminus \{0\}$, satisfies for all $\boldsymbol{\theta} \in \text{supp } Q$,

$$\frac{dP_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_1}^{(P_{\boldsymbol{\theta}|\mathbf{Z}=\mathbf{z}_2}^{(Q,\lambda_2)},\lambda_1)}}{dQ}(\boldsymbol{\theta}) = \frac{\exp\left(\frac{-1}{\lambda_1} L(\mathbf{z}_1, \boldsymbol{\theta}) + \frac{-1}{\lambda_2} L(\mathbf{z}_2, \boldsymbol{\theta})\right)}{\int \exp\left(\frac{-1}{\lambda_1} L(\mathbf{z}_1, \boldsymbol{\nu}) + \frac{-1}{\lambda_2} L(\mathbf{z}_2, \boldsymbol{\nu})\right) dQ(\boldsymbol{\nu})}.$$

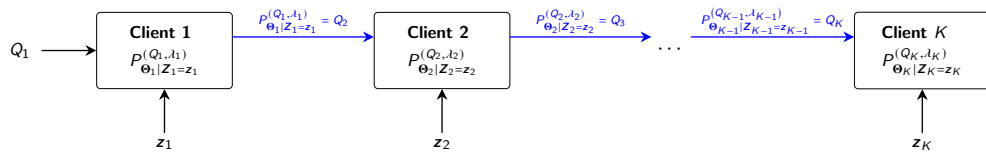
Centralized Performance Guarantees



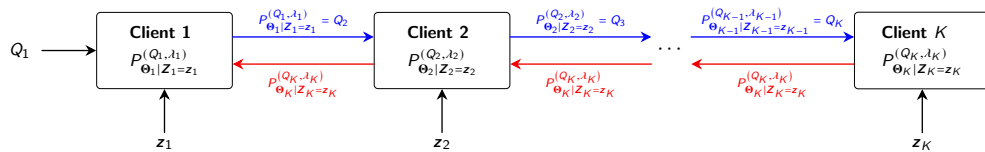
Centralized Performance Guarantees



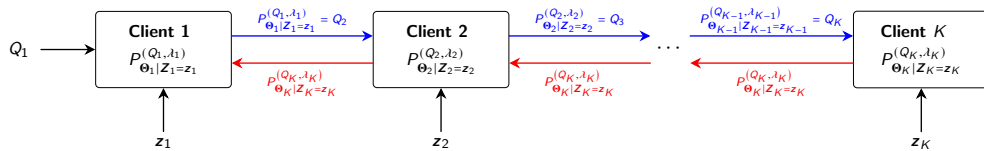
Centralized Performance Guarantees



Centralized Performance Guarantees



Centralized Performance Guarantees



Theorem

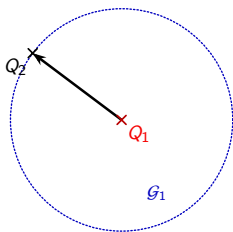
Assume that, for all $k \in \{1, \dots, K\}$, $\lambda_k = \frac{n_0}{n_k} \lambda_0$, for some $\lambda_0 > 0$. Consider some measures $Q_1, Q_2, \dots, Q_K$ satisfying

$$Q_k = \begin{cases} Q_1 & \text{if } k = 1 \\ P^{(Q_{k-1}, \lambda_{k-1})}_{\Theta_{k-1}|Z_{k-1}=z_{k-1}} & \text{if } k \geq 2. \end{cases}$$

Then, for all $\theta \in \text{supp } Q_1$, it follows that,

$$\frac{dP^{(Q_K, \lambda_K)}_{\Theta_K|Z_K=z_K}}{dP^{(Q_1, \lambda_0)}_{\Theta_K|Z=z_0}}(\theta) = 1.$$

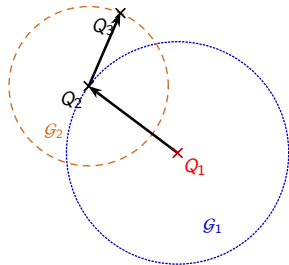
Centralized Performance Guarantees



$$Q_k = \begin{cases} Q_1 & \text{if } k = 1 \\ P_{\Theta_{k-1} | \mathbf{Z}_{k-1} = \mathbf{z}_{k-1}}^{(Q_{k-1}, \lambda_{k-1})} & \text{if } k \geq 2. \end{cases}$$

$$\mathcal{G}_k \triangleq \left\{ P \in \Delta_{Q_1}(\mathcal{M}_k) : D(P \parallel Q_k) \leq D\left(P_{\Theta_k | \mathbf{Z}_k = \mathbf{z}_k}^{(Q_k, \lambda_k)} \parallel Q_k\right) \right\};$$

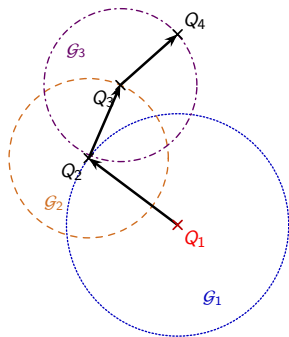
Centralized Performance Guarantees



$$Q_k = \begin{cases} Q_1 & \text{if } k = 1 \\ P_{\Theta_{k-1} | \mathbf{Z}_{k-1} = \mathbf{z}_{k-1}}^{(Q_{k-1}, \lambda_{k-1})} & \text{if } k \geq 2. \end{cases}$$

$$\mathcal{G}_k \triangleq \left\{ P \in \Delta_{Q_1}(\mathcal{M}_k) : D(P \parallel Q_k) \leq D\left(P_{\Theta_k | \mathbf{Z}_k = \mathbf{z}_k}^{(Q_k, \lambda_k)} \parallel Q_k\right) \right\};$$

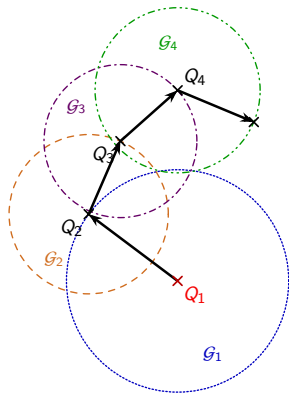
Centralized Performance Guarantees



$$Q_k = \begin{cases} Q_1 & \text{if } k = 1 \\ P_{\Theta_{k-1} | \mathbf{Z}_{k-1} = \mathbf{z}_{k-1}}^{(Q_{k-1}, \lambda_{k-1})} & \text{if } k \geq 2. \end{cases}$$

$$\mathcal{G}_k \triangleq \left\{ P \in \Delta_{Q_1}(\mathcal{M}_k) : D(P \parallel Q_k) \leq D\left(P_{\Theta_k | \mathbf{Z}_k = \mathbf{z}_k}^{(Q_k, \lambda_k)} \parallel Q_k\right) \right\};$$

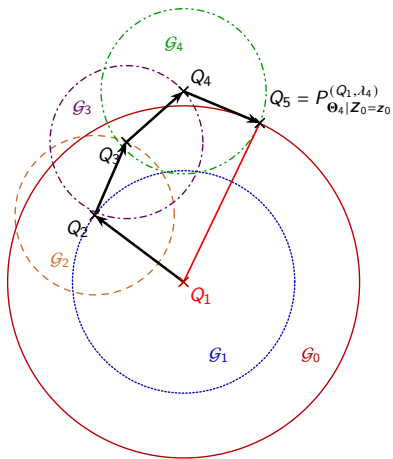
Centralized Performance Guarantees



$$Q_k = \begin{cases} Q_1 & \text{if } k = 1 \\ P_{\Theta_{k-1} | \mathbf{Z}_{k-1} = \mathbf{z}_{k-1}}^{(Q_{k-1}, \lambda_{k-1})} & \text{if } k \geq 2. \end{cases}$$

$$\mathcal{G}_k \triangleq \left\{ P \in \Delta_{Q_1}(\mathcal{M}_k) : D(P \parallel Q_k) \leq D\left(P_{\Theta_k | \mathbf{Z}_k = \mathbf{z}_k}^{(Q_k, \lambda_k)} \parallel Q_k\right) \right\};$$

Centralized Performance Guarantees



$$Q_k = \begin{cases} Q_1 & \text{if } k = 1 \\ P_{\Theta_{k-1} | Z_{k-1} = z_{k-1}}^{(Q_{k-1}, \lambda_{k-1})} & \text{if } k \geq 2. \end{cases}$$

$$\mathcal{G}_k \triangleq \left\{ P \in \Delta_{Q_1}(\mathcal{M}_k) : D(P \parallel Q_k) \leq D\left(P_{\Theta_k | Z_k = z_k}^{(Q_k, \lambda_k)} \parallel Q_k\right) \right\};$$

$$\mathcal{G}_0 \triangleq \left\{ P \in \Delta_{Q_1}(\mathcal{M}_K) : D(P \parallel Q_1) \leq D\left(P_{\Theta_K | Z = z_0}^{(Q_1, \lambda_K)} \parallel Q_1\right) \right\};$$

Thank you for your attention!

Questions/Comments/Typos: yaiza.bermudez@inria.fr

